Motion in Two and Three Dimensional Physics World

Chapter 3

Keeping track of motion in two and three dimensions allows the study of some dynamic objects. A ball thrown in the air will follow a mathematical parabola. The NASA C-9, sometimes called the Vomit Comet, definitely does not follow one dimensional motion. Learning to describe and predict motion in two or more dimensions will be an extension of x-land techniques to y-land and z-land. One application will be the motion of an object thrown into the air moving horizontally and vertically at the same time. The complex motions of objects in a constant gravitational field break down into several one dimensional motions.

Mathematical Tools for Multiple Dimension – Lesson 1

Lesson Objectives

- Use trigonometry to find components of vectors in diagonal directions
- Understand vector notations using unit vectors and matrices
- Apply vector addition, subtraction, dot products, and cross products

Vocabulary

Vector component

Dot product

Cross product

Check Your Understanding

1. What does the vector, $\vec{r} = (5.0m)\hat{x}$, mean?
2. What does the vector, $\vec{r} = (3.0m)\hat{x} + (4.0m)\hat{y}$, mean?
3. Describe the concept of negative velocity.

Introduction

Moving horizontally can be described exclusively with an x-axis. A y-axis from mathematical traditions is a direction perpendicular to the x direction. Introductions to studying motion typically restrict themselves to cases where all movements are limited to one axis at a time. In the real world, it is certainly possible to move diagonally in two or even three dimensions. The goal of this lesson is to learn how vectors can be expressed in multiple dimensions to include ways to treat vectors with special math tools.

Lesson Content

Handling Diagonal-land with Triangles

How do you locate an object in a two dimensional world? In x-land, a simple number with a positive or negative sign would specific an unique location. The answer is that there are several ways. First, you must understand that $3+4=5$. Figure 1 shows how this can true.

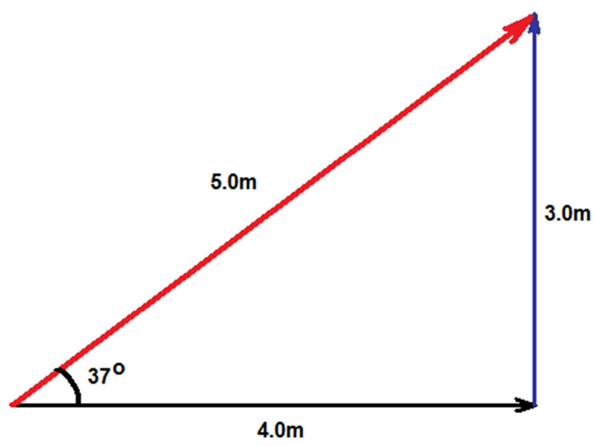


Figure 1. Two dimensional position vector.

Triangles are very useful mathematical tools to deal with the two dimensional physics world. Figure 1 shows a red position vector pointing to the right and up into a diagonal direction. In this case, x-land can not describe this position alone. The vertical direction, or y-land, is needed. For this vector, the red arrow points 5.0m at 37 degrees above the positive x-axis. The measurement of angles is by tradition considered positive measured from the x-axis in the counter-clockwise direction. The principle of similar triangles led to the modern day triangles. Figure 2 shows the triangle in Figure 1 doubled in size. Note that all of the angles stays the same.

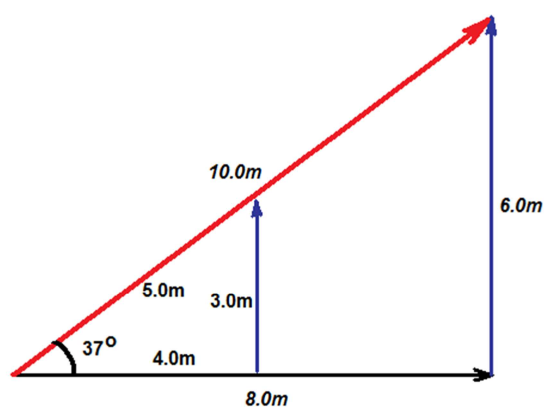
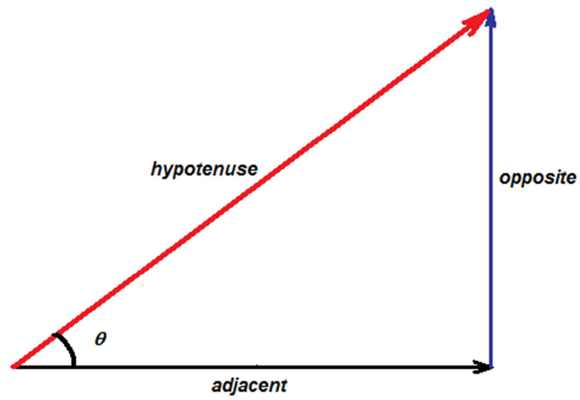


Figure 2. Principle of similar triangles.

The ratio of the sides stays the same no matter how much bigger or smaller the triangle becomes. The definitions of these ratios are very useful and have been conveniently included in all scientific calculators as the trigonometric functions. Table 1 shows the basic trig functions.



Trigonometric Function
$\sin\theta = \dfrac{opposite}{hypotenuse}$
$\cos\theta = \dfrac{adjacent}{hypotenuse}$
$\tan\theta = \dfrac{opposite}{adjacent}$

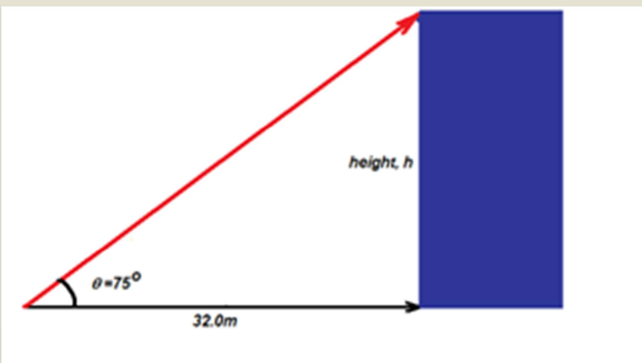
Table 1. Basic Trigonometric Functions

Example – Application of Trigonometry

a) Standing 32.0m from the base of a rectangular building, you measure that the angle from near the ground to the top of the building is 75 degrees. Find the height of the building.

b) In an exercise to measure acceleration on a ramp, you need to measure the angle of the incline with respect to the ground. The ramp is 20.0cm high and the ramp itself is 80.0cm long. Find the angle of the ramp.

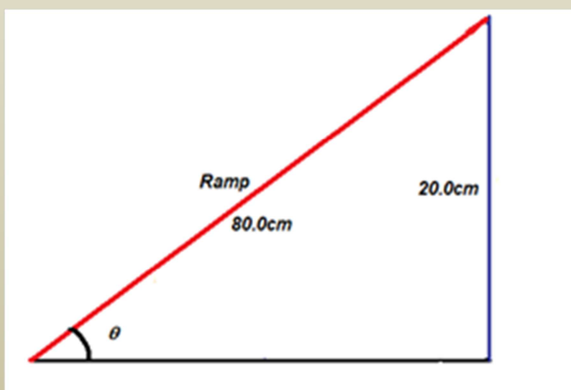
Solution:



a) Label a triangle with knowns and unknowns. Use trig functions to find the unknowns if possible.

$$\tan 75 = \frac{h}{32.0m}$$

Solve for h. $h = (32.0m)\tan 75 = 119m$



b) Label a triangle with knowns and unknowns. $\sin\theta = \frac{20.0}{80.0}$ Take the inverse function of sine to find θ:

$$\theta = Sin^{-1}\left(\frac{20.0}{80.0}\right) = 14.5^{O}$$

Fun ways to express quantities that have a direction

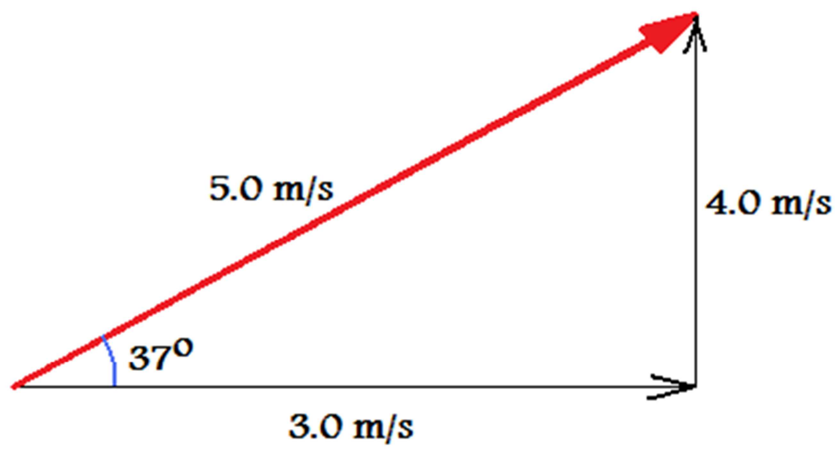
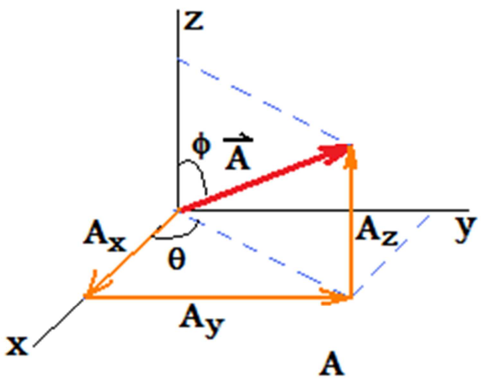


Figure 3. Example of a Two Dimensional Velocity Vector.

If an object moves at 5.0 m/s in a 37 degrees diagonal direction, it can be said that the object is changing its horizontal position and vertical position by 3.0 m/s and 4.0 m/s. In terms of right triangles, sides of 3 and 4 combine to give a hypotenuse of 5 due to the Pythagorean Theorem. The red arrow in figure 3 is a two dimensional vector that can be specified as $\vec{v} = 5.0\frac{m}{s}$ at 37 degrees. Any two dimensional vector can be expressed as a magnitude plus a direction. However, any two dimensional vector can be expressed in terms of its horizontal and vertical components. A **vector component** is the part of a vector that typically lies on a vertical or horizontal axis. The same velocity vector can be written in terms of x and y unit vectors: $\vec{v} = \left(3.0\frac{m}{s}\right)\hat{x} + \left(4.0\frac{m}{s}\right)\hat{y}$. There is another shorter way to express vectors in the unit vector format, using what is called a column matrix. The velocity vector in matrix format would be: $\vec{v} = \begin{pmatrix} 3.0\frac{m}{s} \\ 4.0\frac{m}{s} \end{pmatrix}$. The top number is always the part in x-land or the x component. The next number is always the y component. This system for describing where quantities can point is generalized for three dimensions in Table 2. The three dimensional form of the magnitude plus direction format for expressing vectors can use one magnitude and two angles to specify the location anywhere on a sphere with a radius equal to the magnitude. This system for locating objects is called spherical coordinates.



Vector format	Generic Form
Magnitude + direction (spherical coordinates)	$\vec{A} = \lvert\vec{A}\rvert$ at azimuth angle θ and altitude angle ϕ.
Unit vectors	$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$
Matrices	$\vec{A} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$

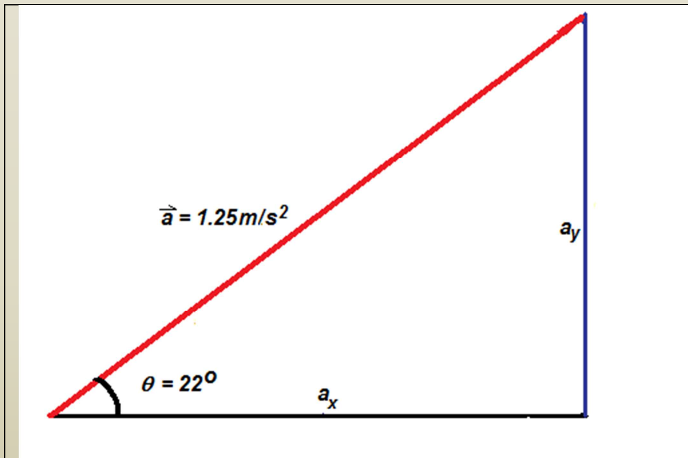
Table 2. Ways to Express a Vector Quantity

Example:

Convert the acceleration vector, $\vec{a} = 1.25\frac{m}{s^2}$ at 22 degrees, into the unit vector form and the matrix form.

Solution:

Draw the vector as a triangle and find the x and y components.



$\sin 22 = \frac{a_y}{1.25\frac{m}{s^2}}$ *and* $a_y = 0.468\frac{m}{s^2}$

$\cos 22 = \frac{a_x}{1.25\frac{m}{s^2}}$ *and* $a_y = 1.16\frac{m}{s^2}$

The acceleration vector in unit vector form is: $\vec{a} = \left(0.468\frac{m}{s^2}\right)\hat{x} + \left(1.16\frac{m}{s^2}\right)\hat{y}$.

The matrix form of the acceleration vector is: $\vec{a} = \begin{pmatrix} 0.468\frac{m}{s^2} \\ 1.16\frac{m}{s^2} \end{pmatrix}$.

Addition, Subtraction, and Special Multiplication for Vectors

Addition and Subtraction of Vectors

Vectors can be combined easiest by writing them in their component form. The general rule for vector addition is:

$$\vec{A} + \vec{B} = (A_x + B_x)\hat{x} + (A_y + B_y)\hat{y} + (A_z + B_z)\hat{z}\ .$$

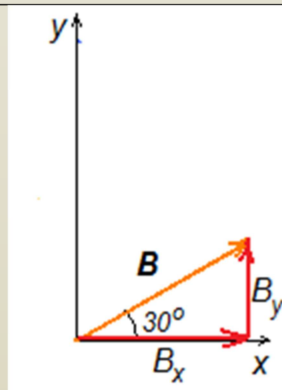
Consider a two dimensional example.

Example: **2D vector addition**

Find the vector resultant from adding vector A and B defined as: $\vec{A} = 15m$ at 90°, and $\vec{B} = 10m$ at 30°.

Solution:

First, express each vector in component form. Use the x and y axis to form a right triangle and find the legs of that triangle

Vector A is already on the y-axis so it's x and y components are known. $\vec{A} = (15.0m)\hat{y}$	 *Vector B becomes a right triangle with the red component vectors. Using trig, vector B becomes:* $\vec{B} = (8.66m)\hat{x} + (5.00m)\hat{y}$	*Adding the vectors in component form is just adding the components.* $\vec{A} + \vec{B} = (0 + 8.66m)\hat{x} + (15.0m + 5.00m)\hat{y}$ $\vec{A} + \vec{B} = (8.66m)\hat{x} + (20.0m)\hat{y}$ *To convert this to a magnitude and direction, use some trig.* $\left\|\vec{A} + \vec{B}\right\| = \sqrt{(8.66m)^2 + (20.0m)^2} = 21.8m$ $\tan\theta = \frac{20.0}{8.66}$ → $\theta = Tan^{-1}\left(\frac{20.0}{8.66}\right) = 67^o$

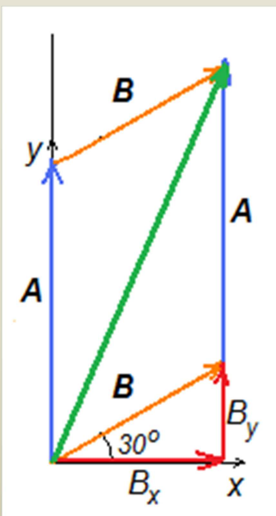
Vector Addition – the Graphical Method

A visual way of adding vectors is to draw them tail to head for as many vectors that are added and show the resultant vector by drawing a vector from the origin to the head of the last vector. The graphical addition of vector A and B are shown in the example below.

Example – Graphically Adding Vectors

Add the vectors A and B graphically.

Solution:



The green arrow shows the result of adding vector A to vector B. To determine a numerical result, a ruler and protractor would help find a magnitude and a direction. Notice the right triangle formed by the x component of B and the sum of the y components. This illustrates the analytical method used in the example above this one.

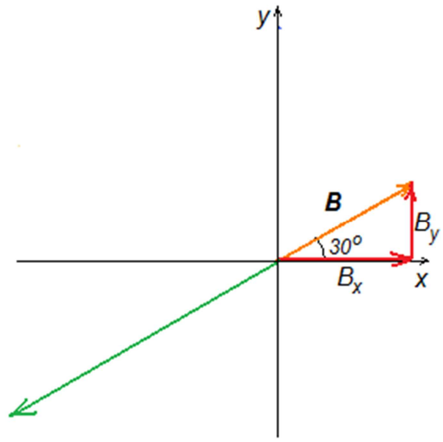
Vector Subtraction

Vector subtraction is the same as addition except with negative components. A negative sign in front of a vector means that all of its components will flip direction by 180 degrees.

Scalar multiplication of a vector only changes its magnitude, and not its direction.

Example:

Describe the vector scalar product, $-2\vec{B}$.



Solution:

The graphical results show that the orange B vector is double in size and in the opposite direction. $-2\vec{B}=2(8.66m)\hat{x}+2(5.00m)\hat{y}$

$-2\vec{B}=(17.3m)\hat{x}+(10.0m)\hat{y}$

In magnitude-direction form,

$-2\vec{B}=20.0m$ *at 210 degrees.*

Special Multiplication of Vectors

Multiplying with numbers is straightforward enough. However, just as the example where 3+4 was not equal to 7, vectors are conceptually different. There are two forms of vector multiplication that will show up in physics world. One is called the dot product and the other is the cross product.

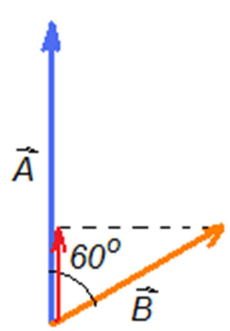
The Dot Product

A dot product is a scalar quantity that represents the product of the part of one vector that is parallel to another vector with that second vector. The dot product of a horizontal vector with a vertical vector would be zero, since no part of those vectors point in the direction of each other. There are two ways to calculate dot products. The two methods are defined and demonstrated in the examples that follow.

Example – Dot product with magnitudes and directions

What is the dot product of a vector A and B defined as: $\vec{A}=15m$ at 90°, and $\vec{B}=10m$ at 30°?

Solution:



The dot product for vectors expressed as magnitudes plus direction is:

$\vec{A}\cdot\vec{B}=|\vec{A}||\vec{B}|\cos\theta_{A/B}$,

where $\theta_{A/B}$ represents the angle between vectors A and B. The angle between the two vectors in this example is shown in the figure to the left.

The red arrow in the figure represents the part of vector B that points in the same direction as vector A. Since the red arrow is the adjacent side of the right triangle formed by vector B, you can see the reason for the cosine function.

$\vec{A}\cdot\vec{B}=|15.0m||10.0m|\cos 60=75m^2$

Example – Dot product with unit vector or matrix vector format

Take the same vectors as before, but express them as unit vectors and matrix vectors.

Unit vector format	Matrix format
$\vec{A}=(15.0m)\,\hat{y}$ $\vec{B}=(8.66m)\,\hat{x}+(5.00m)\,\hat{y}$	$\vec{A}=\begin{pmatrix}0\\15.0m\end{pmatrix}$ $\vec{B}=\begin{pmatrix}8.66m\\5.00m\end{pmatrix}$

The dot product in component forms like unit vector and matrix format can be expressed generically for two arbitrary vectors A and B as:

$\vec{A}\cdot\vec{B}=A_xB_x+A_yB_y+A_zB_z$ (vector component definition of the dot product)

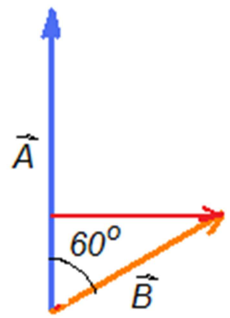
In this example, $\vec{A}\cdot\vec{B}=(0)(8.66m)+(15.0m)(5.00m)+(0)(0)=75.0m^2$

The Cross Product

How much does one vector cross another? The cross product is the opposite of the dot product as it is the product of the parts of two vectors that are perpendicular to each other. A key difference between a dot product and cross product is that the answer is a vector that points in a direction perpendicular to both of the vectors in the cross product. This vector property is useful for a lot of physical concepts that will be seen with later physics principles. For now, consider the mathematics of finding the cross product of two vectors. As with the dot product, there are two ways to find a cross product. An example using the same two vectors A and B will be used to explain the two methods.

Example – Cross product with magnitudes and directions

What is the cross product of a vector A and B defined as: $\vec{A} = 15m$ at 90°, and $\vec{B} = 10m$ at 30°?



Solution:

The cross product for vectors expressed as magnitudes plus direction is:

$$\vec{A} \times \vec{B} = |\vec{A}||\vec{B}|\sin\theta_{A/B},$$

where $\theta_{A/B}$ represents the angle between vectors A and B. The red arrow in the figure represents the part of vector B that points perpendicular to vector A. Since the red arrow is the opposite side of the right triangle formed by vector B, you can see the reason for the sine function.

$$\vec{A} \times \vec{B} = |15.0m||10.0m|\sin 60 = -\left(130m^2\right)\hat{z}$$

To determine the direction of the cross product vector, use the right hand rule. This rule says you should point your outstretched fingers in the direction of the first vector, and then bend your fingers to point in the direction of the second vector. It may be necessary to flip your hand around so you can physically bend your fingers. The direction of your thumb will show the direction of the cross product. Figure 4 shows the right hand rule being implemented.

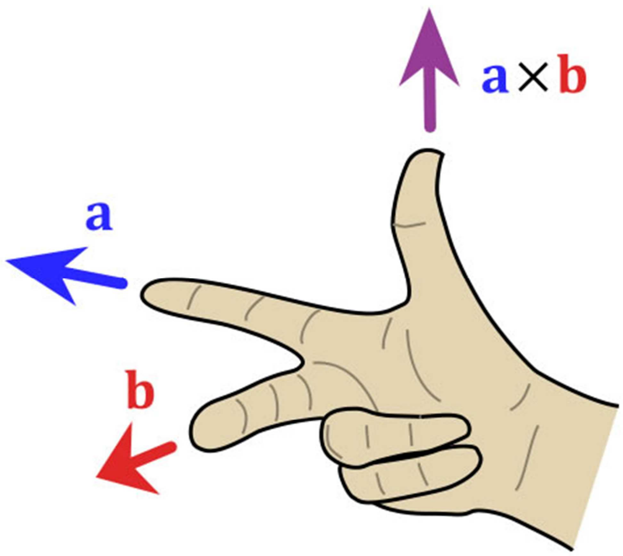


Figure 4. Implementation of the Right Hand Rule <<replace this graphic>>

For this example, your fingers would point up along the y-axis and you would have to flip your hand over to bend them toward the direction of vector B. Your thumb point downward or into the page. By convention, out of the page is considered the +z direction and into the page is the –z direction.

Example – Cross product with unit vector or matrix vector format

The cross product in component form is more complex looking than the magnitude-direction method. The cross product is a determinant of the two vectors. Without experience with matrix algebra, this method may end up being a strange formula to use. The matrix experience will come in later chapters, if that is the case.

The cross product of two generic vectors A and B can be expressed:

$$\vec{A}\times\vec{B}=\left(A_yB_z-A_zB_y\right)\hat{x}+\left(A_zB_x-A_xB_z\right)\hat{y}+\left(A_xB_y-A_yB_x\right)\hat{z}\,.$$

In this example,

$$\vec{A}\times\vec{B}=\left((15.0m)0-0(5.0m)\right)\hat{x}+\left(0(8.66m)-(0)0\right)\hat{y}+\left((0)(5.00m)-(15.0m)(8.66m)\right)\hat{z}=-(130m)\hat{z}$$

Lesson Summary

- Vectors can be written as magnitudes with directions and also in terms of the directional components.

- Addition and subtraction of vectors can be easily managed in the component form and using rules of triangles to convert into magnitudes with directions.
- Vectors can be multiplied using one of two forms: the dot product and cross product. The dot product multiplies the parts of two vectors pointing in the same direction, and the cross product multiplies the parts of two vectors that are perpendicular.

Review Questions

Review Problems

Further Reading / Supplemental Links

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Points to Consider

1. How can unit vector notation be useful to describe the position of an object in three dimensions?
2. In one dimensional motion, the relationship between velocity and position can be written as $v = \frac{dx}{dt}$. What is the definition in multiple dimensions?

Kinematics in Multiple Dimensions – Lesson 2

Lesson Objectives

- Describe the motion of objects in more than one dimension with uniform acceleration
- Predict the motion of objects with varying accelerations in multiple dimensions

Check Your Understanding

1. How can measurements of acceleration and velocity to predict future positions?
2. When is calculus helpful to describe the motion of an object?
3. What is the difference between a vector written as a magnitude with direction and the unit vector format?

Introduction

It makes sense to first study the concepts of position, velocity, and acceleration for objects confined to moving on a line. This one dimensional motion is a simple case that allows a smooth conceptual introduction, but is limited in practical applications. Two dimensional motion will expand your conceptual understanding to flat planes. The level of complication is managed very well by just keeping track of the multi-dimensional motion with separate equations of motion.

Lesson Content

Moving in Diagonal Land with Uniform Acceleration

In x-land, the equations of motion when acceleration is constant were expressed as:

$$x = x_0 + v_0 t + \tfrac{1}{2} a t^2 \qquad v = v_0 + at \qquad v^2 = v_0^{\,2} + 2a\Delta x$$

The way to describe the motion of an object moving with uniform acceleration is to keep track of each coordinate with a separate set of equations. In three dimensions, the equations of motion can be written as:

$$x = x_0 + v_{0x} t + \tfrac{1}{2} a_x t^2 \qquad v_x = v_{0x} + a_x t \qquad v_x^{\,2} = v_{0x}^{\,2} + 2a_x \Delta x$$

$$y = y_0 + v_{0y} t + \tfrac{1}{2} a_y t^2 \qquad v_y = v_{0y} + a_y t \qquad v_y^{\,2} = v_{0y}^{\,2} + 2a_y \Delta y$$

$$z = z_0 + v_{0z} t + \tfrac{1}{2} a_z t^2 \qquad v_z = v_{0z} + a_z t \qquad v_z^{\,2} = v_{0z}^{\,2} + 2a_z \Delta z$$

In order to make this strategy work, the equations only use information specific to their direction. Any vector expressed as a magnitude with direction must be converted into individual components. Note how the three sets of equations for x, y, and z land all have subscripts or actual x, y, and z labels. The exception is time. Time is the same for all directions since they all exist simultaneously.

The equations of motion for constant accelerations can also be written in unit vector format.

$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 + v_{0x}t + \frac{1}{2}a_x t^2 \\ y_0 + v_{0y}t + \frac{1}{2}a_y t^2 \\ z_0 + v_{0z}t + \frac{1}{2}a_z t^2 \end{pmatrix} \qquad \vec{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} v_{0x} + a_x t \\ v_{0y} + a_y t \\ v_{0z} + a_z t \end{pmatrix}$$

There is not anything new to see here except for more detail to manage.

Example:

A freshman who is lost on campus has a motion that can be described with the following position vector:

$$\vec{r}_{freshman} = \begin{pmatrix} x_{freshman} \\ y_{freshman} \\ z_{freshman} \end{pmatrix} = \begin{pmatrix} 2.2m + \left(1.1\tfrac{m}{s}\right)t - \left(0.25\tfrac{m}{s^2}\right)t^2 \\ 8.5m - \left(2.1\tfrac{m}{s}\right)t + \left(0.15\tfrac{m}{s^2}\right)t^2 \\ 1.6m \end{pmatrix}$$

If a helpful senior wants to intercept the student at a time, t = 12.5 seconds, find the constant velocity vector that would make that happen. Assume the senior starts at the origin with the same height, 1.6m.

Solution:

First, find the location of the freshman at t = 12.5s.

$$\vec{r}_{freshman} = \begin{pmatrix} x_{freshman} \\ y_{freshman} \\ z_{freshman} \end{pmatrix} = \begin{pmatrix} 2.2m + \left(1.1\tfrac{m}{s}\right)(12.5s) - \left(0.25\tfrac{m}{s^2}\right)(12.5s)^2 \\ 8.5m - \left(2.1\tfrac{m}{s}\right)(12.5s) + \left(0.15\tfrac{m}{s^2}\right)(12.5s)^2 \\ 1.6m \end{pmatrix} = \begin{pmatrix} -23.1m \\ 5.7m \\ 1.6m \end{pmatrix}$$

To keep track of a second object, you need another set of position vectors.

$$\vec{r}_{senior} = \begin{pmatrix} x_{senior} \\ y_{senior} \\ z_{senior} \end{pmatrix} = \begin{pmatrix} v_x t \\ v_y t \\ 1.6m + v_{0z} t \end{pmatrix}$$

. Plug in t = 12.5s to the left side and set the position vector equal to the location of the freshman on the right side.

$$\vec{r}_{senior} = \begin{pmatrix} -23.1m \\ 5.7m \\ 1.6m \end{pmatrix} = \begin{pmatrix} v_x (12.5s) \\ v_y (12.5s) \\ 1.6m + v_{0z} (12.5s) \end{pmatrix}$$

. Solving for the velocities,

$$\vec{v}_{senior} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} -1.85 \\ 0.46 \\ 0 \end{pmatrix} \left[\tfrac{m}{s}\right]$$

. Only in physics world would someone have such oddly specific measurements of a wandering freshman, but the predictive power of physics is demonstrated.

Accelerations that depend on time

If the acceleration is anything but a constant, the kinematic equations of motion will not apply and calculus must be used. The relationship between position, velocity, and acceleration is the same as it was for one-dimensional motion.

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$$

Each of these components expressed as derivatives are the instantaneous velocities in their respective directions.

$$v_x = \frac{dx}{dt} \qquad v_y = \frac{dy}{dt} \qquad v_z = \frac{dz}{dt} \qquad \text{and} \qquad a_x = \frac{dv_x}{dt} \qquad a_y = \frac{dv_y}{dt} \qquad a_z = \frac{dv_z}{dt}$$

Integration can be used as the anti-derivative to find position from velocity and velocity from acceleration.

$$\vec{r} = \int \vec{v} \cdot dt \qquad \vec{v} = \int \vec{a} \cdot dt$$

These vectors can be expressed in component form, which reveals three identical relationships.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \int \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \cdot dt \qquad \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \int \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \cdot dt$$

This matrix notation can be expanded to show the individual components.

$x = \int v_x \cdot dt \quad y = \int v_y \cdot dt \quad z = \int v_z \cdot dt$ and $v_x = \int a_x \cdot dt \quad v_y = \int a_y \cdot dt \quad v_z = \int a_z \cdot dt$

The calculus tools will allow you to find the functional relationships between position, velocity, and acceleration.

Example:

The location of a wandering pet cat is monitored by GPS radio telemetry. The cat is constantly accelerating in what is defined as the x-direction, but the acceleration in the y-direction starts with no acceleration and is linearly increasing. Initially, the cat is located at $\vec{r}_0 = \begin{pmatrix} 8.00 \\ -2.00 \end{pmatrix} [m]$ with an initial velocity, $\vec{v}_0 = \begin{pmatrix} 1.00 \\ 0.500 \end{pmatrix} \left[\frac{m}{s}\right]$ at t = 0.00s. Later, at t = 4.00s, the cat has a position vector equal to $\vec{r}_1 = \begin{pmatrix} -14.0 \\ 10.0 \end{pmatrix} [m]$. Find an expression with respect to time for the acceleration, velocity, and position of the cat.

Solution:

The cat moves with a constant acceleration in the x-direction, so the kinematic equations can be used.

$x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2$ → $-14.0m = 8.00m + \left(1.00\frac{m}{s}\right)(4.00s) + \frac{1}{2}a_x (4.00s)^2$

$a_x = 3.25\frac{m}{s^2}$.

The cat is said to move with a linearly increasing acceleration in the y-direction, so $a_y = kt$. The k is some constant since the rate of increase is not known. The acceleration function can be integrated once to get a velocity equation, and again to find position with respect to time.

$v_y = \int a_y \cdot dt = \int kt \cdot dt = \frac{k}{2}t^2 + v_{0y}$ where the constant of integration is the initial value of y-velocity. Position can be found by integrating velocity.

$y = \int v_y \cdot dt = \int \left(\tfrac{1}{2}kt^2 + v_{0y}\right)dt = \tfrac{1}{6}kt^3 + v_{0y}t + y_0$. The boundary conditions can now be substituted.

$$y = \tfrac{1}{6}kt^3 + v_{0y}t + y_0 \rightarrow \quad 10.0m = \tfrac{1}{6}k(4.00s)^3 + \left(0.500\tfrac{m}{s}\right)(4.00s) - 2.00m \rightarrow$$
$$k = 1.31\tfrac{m}{s^3}$$

All of these boundary conditions and the k-constant can be filled into the equations for x and y to construct the vectors.

$$x = 8.00m + \left(1.00\tfrac{m}{s}\right)t + \left(1.63\tfrac{m}{s^2}\right)t^2,\ v_x = v_{0x} + a_x t \rightarrow v_x = 1.00\tfrac{m}{s} + \left(3.25\tfrac{m}{s^2}\right)t,\ a_x = 3.25\tfrac{m}{s^2}$$

$$y = \tfrac{1}{6}kt^3 + v_{0y}t + y_0 \rightarrow y = \left(1.31\tfrac{m}{s^3}\right)t^3 + \left(0.500\tfrac{m}{s}\right)t - 2.00m,\ v_y = \frac{k}{2}t^2 + v_{0y} \rightarrow$$

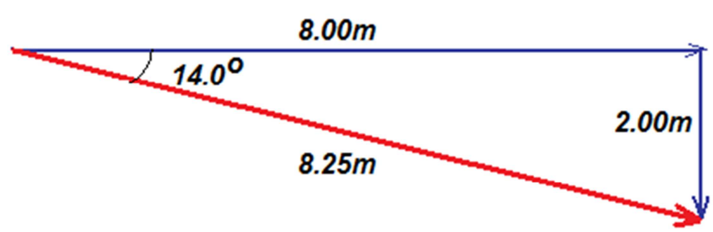
$v_y = \left(0.656\tfrac{m}{s^3}\right)t^2 + 0.500\tfrac{m}{s}$, and $a_y = \left(1.31\tfrac{m}{s^3}\right)t$. These can be organized in matrix notation.

$$\vec{r} = \begin{pmatrix} 8.00m + \left(1.00\tfrac{m}{s}\right)t + \left(1.63\tfrac{m}{s^2}\right)t^2 \\ \left(1.31\tfrac{m}{s^3}\right)t^3 + \left(0.500\tfrac{m}{s}\right)t - 2.00m \end{pmatrix},\ \vec{v} = \begin{pmatrix} 1.00\tfrac{m}{s} + \left(3.25\tfrac{m}{s^2}\right)t \\ \left(0.656\tfrac{m}{s^3}\right)t^2 + 0.500\tfrac{m}{s} \end{pmatrix},\ \text{and}\ \vec{a} = \begin{pmatrix} 3.25\tfrac{m}{s^2} \\ \left(1.31\tfrac{m}{s^3}\right)t \end{pmatrix}.$$

Using the magnitude-direction vector format in solutions

It is common for information to be given in the form of a magnitude with a direction in degrees. To use the information in the component equations, the magnitude and direction must be broken into their respective parts.

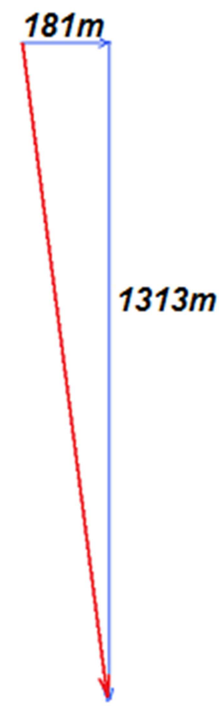
Consider the example in the previous section about the cat. The initial position of the cat could have been given as a magnitude plus a direction. Figure 5 shows how the Pythagorean theorem is applied to find the magnitude. The angle for the direction is found using a trigonometric function and taking its inverse.



Example:

Find the position of the cat at t = 10.0s as a magnitude and direction.

Solution:



First, the components can be found using the individual equations in the component vector format.

$$\vec{r} = \begin{pmatrix} 8.00m + \left(1.00\tfrac{m}{s}\right)t + \left(1.63\tfrac{m}{s^2}\right)t^2 \\ \left(1.31\tfrac{m}{s^3}\right)t^3 + \left(0.500\tfrac{m}{s}\right)t - 2.00m \end{pmatrix}$$ *At t = 10.0s,*

$$\vec{r} = \begin{pmatrix} 8.00m + \left(1.00\tfrac{m}{s}\right)(10.0s) + \left(1.63\tfrac{m}{s^2}\right)(10.0s)^2 \\ \left(1.31\tfrac{m}{s^3}\right)(10.0s)^3 + \left(0.500\tfrac{m}{s}\right)(10.0s) - 2.00m \end{pmatrix} = \begin{pmatrix} 181m \\ 1313m \end{pmatrix}$$

The components form a right triangle. The magnitude is the hypotenuse and the angle with the x-axis is the direction for this position vector at 10.0s. $|\vec{r}| = \sqrt{r_x^{\,2} + r_y^{\,2}} = \sqrt{(181m)^2 + (1313m)^2} = 1325m +$

$$\tan\theta = \frac{-1313m}{181m} \rightarrow \theta = -82^0$$

Lesson Summary

- Motion in multiple dimensions is the same as one dimensional motion in terms of relationships
- Two and three dimensional motion requires managing information in terms of components

Review Questions

Review Problems

Further Reading / Supplemental Links

Points to Consider

1. If an object is thrown at an angle in the air, what is the acceleration vector?
2. Will the kinematic equations in the x and y direction for an object in free fall?
3. When is calculus useful for an object thrown in the air?

Projectile Motion – Lesson 3

Lesson Objectives

- Define projectile motion
- Be able to predict the motion of a projectile

Vocabulary

Projectile

Object free of all constraints except the constant acceleration due to gravity

Trajectory

Path of a projectile

Check Your Understanding

1. Explain the terms in the following equations:

$$x = x_0 + v_{0x}t + \tfrac{1}{2}a_x t^2 \qquad v_x = v_{0x} + a_x t \qquad {v_x}^2 = {v_{0x}}^2 + 2a_x \Delta x$$

$$y = y_0 + v_{0y}t + \tfrac{1}{2}a_y t^2 \qquad v_y = v_{0y} + a_y t \qquad {v_y}^2 = {v_{0y}}^2 + 2a_y \Delta y$$

2. When an object thrown upward reaches its highest point, what do you know about its motion?

Introduction

Projectile motion is an ideal application of the knowledge of motion. There is not anything new to learn except for some specifics dealing with a constant acceleration in one direction and none in the other direction. Since gravity is the source of the only acceleration, almost everyone can relate to this application in every day experience.

Lesson Content

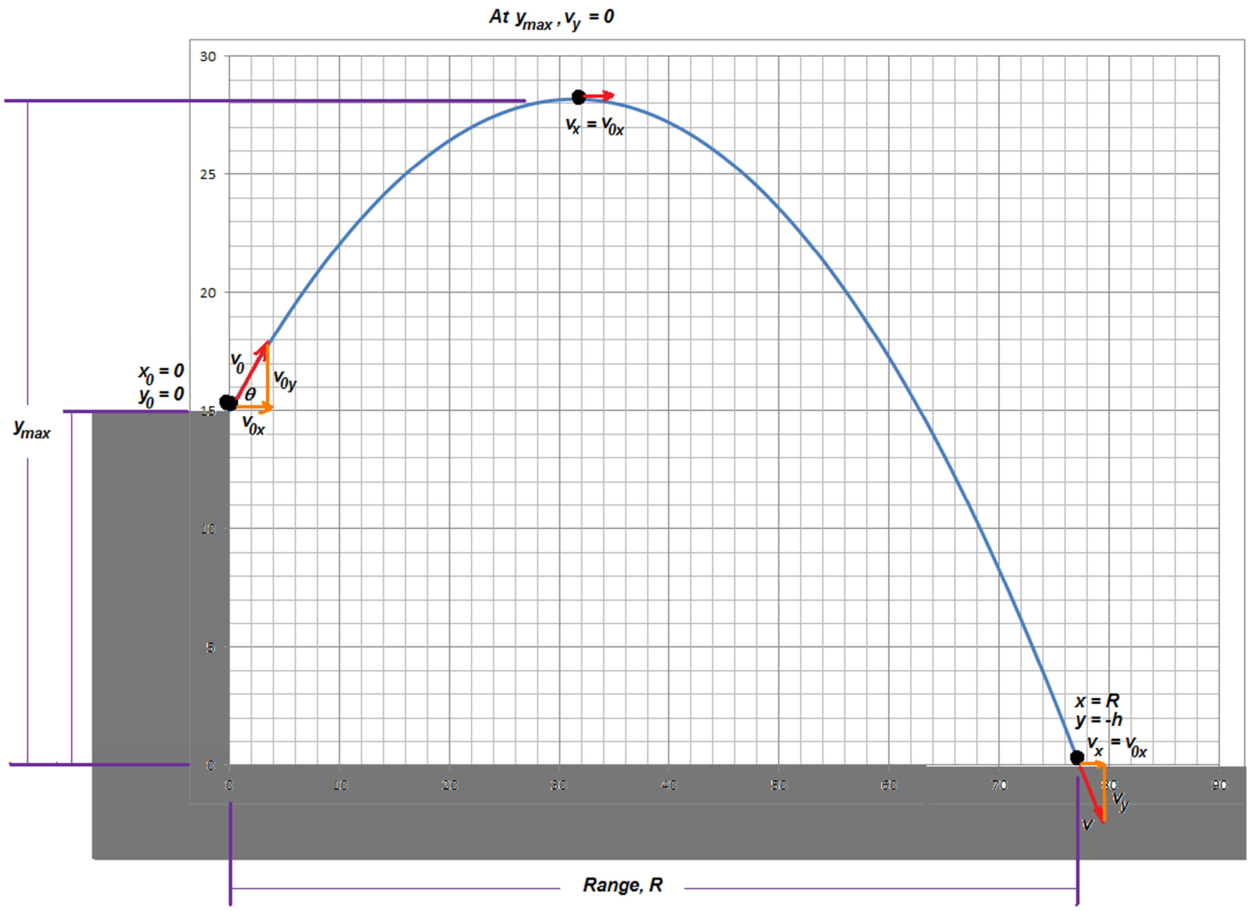


Figure 6, Motion of a projectile.

Figure 6 shows the path of a **projectile** fired diagonally in blue, subject only to the influence of gravity. The path, called a **trajectory**, is a parabola which results from a difference in vertical and horizontal rates of motion. In the case of a projectile, there is a downward acceleration of $g = 9.81\frac{m}{s^2}$ and no acceleration in the horizontal direction. Written as a matrix vector, acceleration due to gravity near the Earth is $a = \begin{pmatrix} a_x \\ a_y \end{pmatrix} = \begin{pmatrix} 0 \\ -g \end{pmatrix}$. Compared to one dimensional motion, projectile motion seems complex. However, if the motion is treated as two separately tracked motions in the x and y directions, it is the same in terms of the application of kinematic equations. Those equations are shown in Table 3, and followed by the substitution of the acceleration vector due to gravity near the surface of the Earth.

Motion in the x-direction

Generic motion	$x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2$	$v_x = v_{0x} + a_x t$	$v_x{}^2 = v_{0x}{}^2 + 2a_x \Delta x$
For projectiles	$x = x_0 + v_{0x}t$	$v_x = v_{0x}$	$v_x{}^2 = v_{0x}{}^2$

Motion in the y-direction

Generic motion	$y = y_0 + v_{0y}t + \frac{1}{2}a_y t^2$	$v_y = v_{0y} + a_y t$	$v_y{}^2 = v_{0y}{}^2 + 2a_y \Delta y$
For projectiles	$y = y_0 + v_{0y}t - \frac{1}{2}gt^2$	$v_y = v_{0y} - gt$	$v_y{}^2 = v_{0y}{}^2 - 2g\Delta y$

Table 3. Application of two dimensional motion for projectiles.

Steps to solve a Projectile motion problem

Table 4 shows some steps to follow to set-up and solve a projectile motion problem.

1. Draw a picture.	Label the known and unknown quantities.
2. Choose an origin.	A typical choice is to set the beginning position as the origin, but the choice is yours.
3. Break velocities into components.	The x-land and y-land equations only work with components, so make the initial velocity into a triangle and find the sides.
4. Solve for unknowns using the equations with chosen start and end positions.	Many students struggle with which equation to use first. Experience may help make that choice, but any of them are fine. Try them all if you must and then use algebra to solve. The same equation may be used more than once if you have different beginning and ending positions.

Example:

A t-shirt gun can fire a t-shirt at a velocity of 38 miles per hour (17.0 m/s) at 45 degrees. The t-shirt gun is fired from the top of a basketball gym toward the floor which is 7.00 m lower. Answer the following about the t-shirt.

a) What is the maximum height of the t-shirt above the gym floor?

b) How much time is the t-shirt in the air?

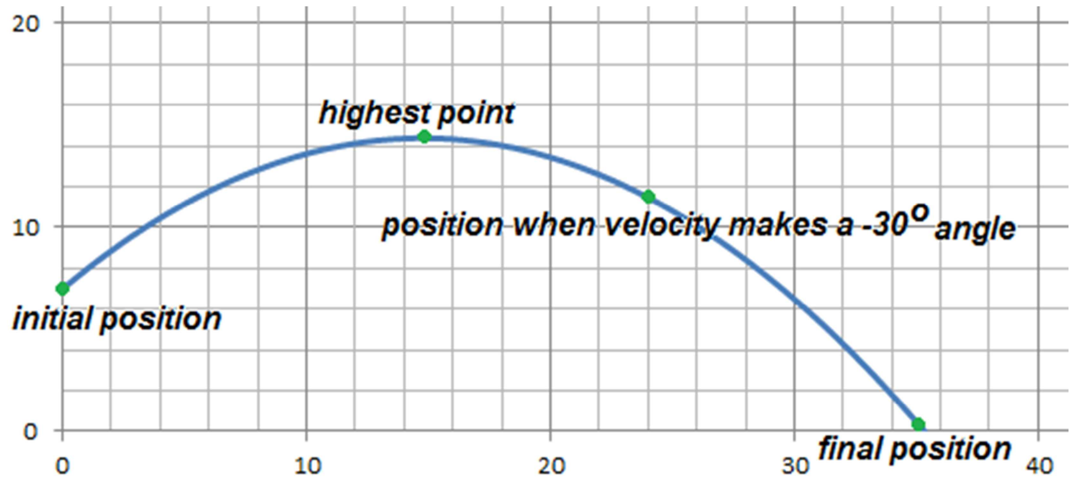
c) What is the range of the t-shirt?

d) What is the final velocity of the t-shirt the moment before hitting the floor?

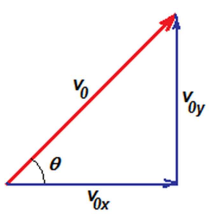
e) Where is the t-shirt located when its velocity is pointing at -30 degrees?

Solution:

Step 1: Draw a picture.



Step 2: Choose an origin. The origin is the gyn floor level beneath where the t-shirt is fired.



Step 3: Break the initial velocity into components.

$$v_{0x} = v_0 \cos\theta = \left(17.0\tfrac{m}{s}\right)\cos 45 = 12.02\tfrac{m}{s}$$

$$v_{0y} = v_0 \sin\theta = \left(17.0\tfrac{m}{s}\right)\sin 45 = 12.02\tfrac{m}{s}$$

Step 4: Solve for unknowns.

Part (a) is looking for the highest point. At the highest point, the y-velocity is zero. Start with the y-velocity equation for projectiles.

$v_y = v_{0y} - gt$ → $0 = 12.02\tfrac{m}{s} - \left(9.81\tfrac{m}{s^2}\right)t$ → $t = 1.23s$ *This is the time it takes to reach the highest point. The y-equation can give the actual height.*

$$y = y_0 + v_{0y}t - \tfrac{1}{2}gt^2 \rightarrow y_{\max} = 7.0m + \left(12.02\tfrac{m}{s}\right)(1.23s) - \left(4.905\tfrac{m}{s^2}\right)(1.23s)^2 \rightarrow \boxed{y_{\max} = 14.4m}$$

Part (b) asks for the total time in the air. This is a new end point, so the same equations can be used with the different end values.

$y = y_0 + v_{0y}t - \tfrac{1}{2}gt^2 \rightarrow 0 = 7.0m + \left(12.02\tfrac{m}{s}\right)t - \left(4.905\tfrac{m}{s^2}\right)t^2$ The quadratic equation can be used to find two answers for time. The logic of the situation requires you to choose the positive solution. There is a way to avoid the quadratic equation by using the two velocity equations.

$$v_y^2 = v_{0y}^2 - 2g\Delta y \rightarrow v_y^2 = \left(12.02\tfrac{m}{s}\right)^2 - 2\left(9.81\tfrac{m}{s^2}\right)(-7.00m) = 281.8\tfrac{m^2}{s^2} \rightarrow v_y = \pm 16.8\tfrac{m}{s}$$

When the t-shirt is hitting the ground, it is moving in the downward vertical direction. This logic is how you can choose between the plus or minus solution from the v_y^2 equation.

$$v_y = v_{0y} - gt \rightarrow -16.8\tfrac{m}{s} = 12.02\tfrac{m}{s} - \left(9.81\tfrac{m}{s^2}\right)t \rightarrow t = 2.94s$$

Part (c) asks for the range, which is the furthest x-value of the t-shirt.

The range is the final x-value. $x = x_0 + v_{0x}t \rightarrow R = 0 + \left(12.02\tfrac{m}{s}\right)(2.94s) \rightarrow$

$\boxed{R = 35.3m}$

Part (d) asks for the final velocity. The kinematic equations only know how to deliver the components, but those can be turned into a magnitude and direction easily enough.

$$v_x = v_{0x} = 12.02\tfrac{m}{s} \qquad v_y = -16.8\tfrac{m}{s} \rightarrow |\vec{v}| = \sqrt{v_x^2 + v_y^2} = \sqrt{\left(12.02\tfrac{m}{s}\right)^2 + \left(-16.8\tfrac{m}{s}\right)^2} = 20.7\tfrac{m}{s}$$

The angle can be found using trigonometric functions.

$$\tan\theta = \frac{v_y}{v_x} = \frac{-16.8}{12.02} \rightarrow \theta = Tan^{-1}(\tan\theta) = Tan^{-1}(-1.40) = -54^0.$$

The final velocity is $\boxed{20.7\frac{m}{s}}$ at $\boxed{-54^0}$.

Part (e) introduced another end point somewhere on the trajectory coming downward.

Just as in part (d), the final velocity must be found in parts. The x-velocity never changes. $v_x = v_{0x} = 12.02\frac{m}{s}$. If the final angle is -30 degrees, the tangent function can be used to find the y-velocity. $\tan(-30) = \frac{v_y}{12.02\frac{m}{s}} \rightarrow v_y = -6.94\frac{m}{s}$. And, the time to reach this velocity is described by the y-velocity equation. $v_y = v_{0y} - gt \rightarrow$ $-6.94\frac{m}{s} = 12.02\frac{m}{s} - \left(9.81\frac{m}{s^2}\right)t \rightarrow t = 1.93s$

The x and y equations will reveal the location. $x = x_0 + v_{0x}t \rightarrow$ $x = 0 + \left(12.02\frac{m}{s}\right)(1.93s) = 23.2m$

$y = y_0 + v_{0y}t - \frac{1}{2}gt^2 \rightarrow y = 7.0m + \left(12.02\frac{m}{s}\right)(1.93s) - \left(4.905\frac{m}{s^2}\right)(1.93s)^2 = 11.9m$

The final position can be expressed as a matrix vector: $\boxed{\vec{r} = \begin{pmatrix} 23.2m \\ 11.9m \end{pmatrix}}$ Or, the position can be described as a magnitude and direction from the origin: 26.1m at 27 degrees.

Lesson Summary

- Projectile Motion is an application of multidimensional motion.
- A projectile has zero vertical velocity at its highest point.
- The range of a projectile is its final horizontal position.

Review Questions

Review Problems

Further Reading / Supplemental Links

Points to Consider

1. Vectors are useful organizational tools for describing motion. What kind of vector operations would be useful to describe the motion of two or more objects?
2. If an airplane is flying into a wind at some direction, how will the direction of the airplane be affected?

Relative Motion – Lesson 4

Lesson Objectives

- Use vector mathematics to predict combined velocities
- Describe perceived motion from particular points of view

Vocabulary

Relative velocity

Velocity of an object described from a particular point of view. The vector definition of relative velocity is $\vec{v}_{B/A} = \vec{v}_B - \vec{v}_A$.

Check Your Understanding

1. Given a vector A and C, and the definition A+B=C, how could you find vector B?
2. Describe the motion of a swimmer trying to cross a river?

Introduction

Motion is defined according to a particular frame of reference. Some frames of reference are moving which changes how concepts like velocity and acceleration are described. Riders on a bus will see cars moving along side moving at slower rates than someone standing on the side of the street looking at those same cars. The descriptions can be reconciled by using vectors with respect to each frame of reference.

Lesson Content

Combined velocities

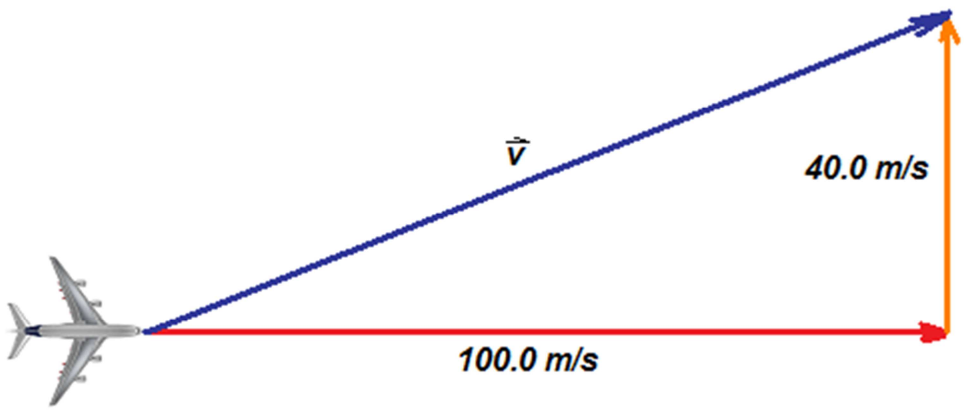
A boat moving in a fast moving river will experience several velocities. The measures of velocity of the boat will depend on whom you ask. A person on the shore of the river will offer a different velocity than someone floating down the river. This confusion can be alleviated by describing each velocity in terms of particular frames of reference and using vector addition and subtraction to relate the velocities.

Example:

An airplane is capable of traveling at 100.0 m/s on a windless day. In this case, the plane flies due east at 100.0 m/s while the wind gives the plane a velocity of 40.0 m/s due north. What is the velocity of the airplane as viewed by someone on the ground?

Solution:

The velocity of the airplane and wind combined is the addition of the airplane velocity to the wind velocity. $\vec{v}_{airplane/wind} = \vec{v}_{airplane} + \vec{v}_{wind}$



Written in vector form: $\vec{v}_{airplane/wind} = \begin{pmatrix} 100.0\frac{m}{s} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 40.0\frac{m}{s} \end{pmatrix} = \begin{pmatrix} 100.0\frac{m}{s} \\ 40.0\frac{m}{s} \end{pmatrix}$. The final velocity can also be expressed as a magnitude (blue arrow in the figure) and direction.

$\boxed{\vec{v}_{airplane/wind} = 108\frac{m}{s}}$ at $\boxed{\theta = 21.8^0}$.

Example:

A boat is capable of traveling 20.0m/s in still water. The boat turns its rudder so that the boat points upstream at an angle of 45 degrees. If the boat ends up at a point 200.0m across the river, find the velocity of the river.

Solution:

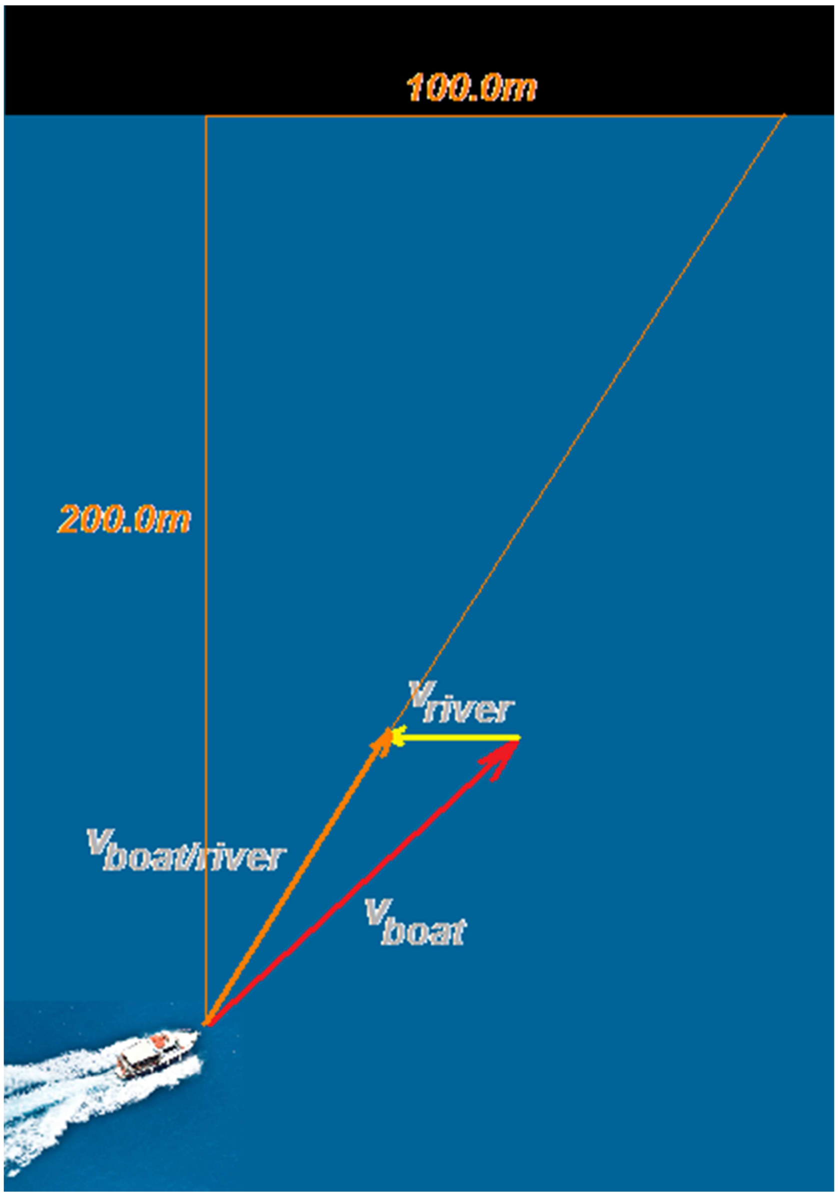
As with any sort of problem, a picture would be useful.

In words, the relationship between the vectors is made clear. The velocity of the boat combined with the velocity of the river yields a velocity of the boat and river combined.

$\vec{v}_{boat} + \vec{v}_{river} = \vec{v}_{boat/river}$. The boat's velocity is given in magnitude-direction form so its components must be found. The river is flowing to the left with no vertical component. The combined velocity is not given but the direction is the same as

the physical triangle describing where the boat travels. The angle can be found with tangent: $\tan\theta = \frac{100}{200}$ → $\theta = 26.6^0$.

$$v_{boat} = \begin{pmatrix} (20.0\frac{m}{s})\cos 45 \\ (20.0\frac{m}{s})\sin 45 \end{pmatrix} = \begin{pmatrix} 14.1\frac{m}{s} \\ 14.1\frac{m}{s} \end{pmatrix},\ \vec{v}_{river} = \begin{pmatrix} -v_{river} \\ 0 \end{pmatrix},\ \text{and}\ v_{boat/river} = \begin{pmatrix} v_{boat/river}\sin 26.6 \\ v_{boat/river}\cos 26.6 \end{pmatrix}.$$



The vector components become equations. The x components create the equation:

$$14.1\tfrac{m}{s} - v_{river} = v_{boat/river}\sin 26.6$$

The y components form the equation:

$$14.1\tfrac{m}{s} + 0 = v_{boat/river}\cos 26.6$$

The y equation reveals the combined boat/river velocity:

$$v_{boat/river} = 15.8\tfrac{m}{s}.$$

This can be plugged into the x equation result.

$$14.1\tfrac{m}{s} - v_{river} = (15.8\tfrac{m}{s})\sin 26.6$$

$$\boxed{v_{river} = 7.04\tfrac{m}{s}}.$$

Velocity as seen from one moving object to another

A frame of reference is a coordinate system established to present a particular point of view. It is possible to have a moving frame of reference to answer questions about the motion of other

objects would appear from a moving object's perspective. In one dimension, cars approaching each other from opposite directions are an example of relative velocities. From the point of view of a passenger in one of the cars, the ground and trees outside could be imagined to be running in the opposite direction of the car while the car was still. An approaching car would have that perceived ground velocity added to the second car's velocity. In equation form, the velocity of car B as seen by car A can be expressed as $\vec{v}_{B/A} = \vec{v}_B - \vec{v}_A$.

Example:

Three cars travel on a two lane highway. Car A travels at 30.0 m/s eastward behind a car B which is traveling at 20.0 m/s. Car C is in the other lane traveling westward at 35.0 m/s.

a) If the driver in car B looks in their rear view mirror, what velocity is car A approaching?

b) Car A is about to pass car B and notices car C approaching. What is the velocity of car C as seen from car A?

Solution:

$\vec{v}_{A/B} = \vec{v}_A - \vec{v}_B = \begin{pmatrix} 30.0\frac{m}{s} \\ 0 \end{pmatrix} - \begin{pmatrix} 20.0\frac{m}{s} \\ 0 \end{pmatrix} = \begin{pmatrix} 10.0\frac{m}{s} \\ 0 \end{pmatrix}$. *The expression $\vec{v}_{A/B}$ reads as the velocity of car A as observed from car B and is equal to 10.0 m/s eastward. From car A's point of view, the answer becomes negative as the car B seems to approach it from the west.*

$\vec{v}_{C/A} = \vec{v}_C - \vec{v}_A = \begin{pmatrix} -35.0\frac{m}{s} \\ 0 \end{pmatrix} - \begin{pmatrix} 30.0\frac{m}{s} \\ 0 \end{pmatrix} = \begin{pmatrix} -65.0\frac{m}{s} \\ 0 \end{pmatrix}$. *When passengers in car A observes car C, they see a combined high speed of 65.0 m/s toward the west.*

Example:

Two jet ski watercraft are traveling on a lake. Jet ski A has a velocity of 20.0 m/s at 45 degrees north of east. Jet ski B is traveling at 10.0 m/s due south. What is the relative speed of B as seen by A?

Solution:

$v_A = \begin{pmatrix} 14.1\frac{m}{s} \\ 14.1\frac{m}{s} \end{pmatrix}$, $v_B = \begin{pmatrix} 0 \\ -10\frac{m}{s} \end{pmatrix}$. *Find* $\vec{v}_{A/B} = \vec{v}_A - \vec{v}_B$. $\vec{v}_{A/B} = \begin{pmatrix} 14.1\frac{m}{s} \\ 14.1\frac{m}{s} \end{pmatrix} - \begin{pmatrix} 0 \\ -10\frac{m}{s} \end{pmatrix} = \begin{pmatrix} 14.1\frac{m}{s} \\ 24.1\frac{m}{s} \end{pmatrix}$. *In magnitude-direction form, the velocity of jet ski A as seen by jet ski B is* $\boxed{27.9\frac{m}{s}}$ *at* $\boxed{60^\circ}$.

Lesson Summary

- Velocity as seen from a stationary point may be different than the velocity from a moving frame of reference.
- If two objects are in motion, the motion relative to another can be described by $\vec{v}_{B/A} = \vec{v}_B - \vec{v}_A$.

Review Questions

Review Problems

Further Reading / Supplemental Links

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Points to Consider

Describing motions in multiple directions or dimensions was made easier to manage using vectors. How can the study of pushes or pulls be made easier to manage with vectors?